

## CHAPTER 5

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# THE ROLE OF TASKS IN PROMOTING DISCOURSE SUPPORTING MATHEMATICAL LEARNING

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In this paper we consider the role that tasks can play in promoting substantive mathematical discourse that leads to opportunities for learning. We will describe a task that we have used successfully in an algebra course for K–12 in-service teachers as part of the Oregon Mathematics Leadership Institutes (OMLI)<sup>1</sup>. An analysis of both the task itself and the participants' mathematical activity as they engaged with the task will be provided with the goal of identifying and articulating: the kinds of mathematical discourse that were generated, the kinds of opportunities for mathematical learning that this discourse provided, and design features of the task that seemed to support the discourse and subsequent learning opportunities.

**KEYWORDS:** promoting discourse, learning, instructional design, tasks, symmetry, justification

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## INTRODUCTION

Recent reform documents, such as those published by the National Council of Teachers of Mathematics (NCTM, 1991, 2000) have called for a shift from a focus primarily on procedural knowledge of mathematics to one that includes conceptual understanding. Making such a shift requires rethinking the learning goals we set for our students and how students can best achieve these goals (Hiebert, 2003). This means classrooms are shifting from systems where teachers are the purveyors of knowledge, to systems in which students take an active role in their own learning. To that end, communication is becoming an increasingly important component of classroom practice and researchers have become increasingly interested in studying the discourse that occurs in classrooms (Ilaria, 2002).

Students can learn about what it means to do mathematics by engaging in activities such as questioning, challenging, and justifying (Stein, 2007). They can also construct meanings for mathematical ideas (NCTM, 2000). For instance, by reflecting and building on mathematical explanations of their peers, students can reorganize their ideas and make connections between ideas (Forman, 2003; Hatano & Inagaki, 1991; Lampert, 1986). Additionally, as students communicate their reasoning, they can develop more sophisticated understandings of mathematical ideas (Simon & Blume, 1996). For example, as they communicate and justify conjectures, students engage in generalization and abstraction, processes that lead to more sophisticated mathematical understandings (Cobb, Boufi, McClain, & Whitenack, 1997; Lampert & Cobb, 2003).

Generating productive discussions requires more than simply asking students to talk. The mathematical discourse is shaped by the instructional tasks and by how the discussions are facilitated (Himmelberger & Schwartz, 2007). Since the quality of discourse affects the potential for mathematical understanding (Kazemi & Stipek, 2001), it is important to use tasks that can move students toward precision, clarification, and generalization (Himmelberger & Schwartz, 2007). Further, the discourse needs to be centered on meaningful mathematical ideas and be based on the ways students come to know those ideas (Kysh, Thompson, & Vicinus, 2007). It is important to ensure that all students participate in the discussions (e.g. Stein, 2007), and that students' contributions to discussions go beyond their personal ideas and connect to the ideas of others (Staples & Colonis, 2007).

In sum, discourse is seen to have the potential to play an important role in the learning of mathematics. However, it is important that the conversations focus on significant mathematical ideas and that the nature of the students' discourse supports the learning of these ideas. Furthermore, to support learning for *all* students, it is important that *all* students are active participants in the classroom discourse.

In this paper we consider the role that tasks can play in promoting substantive mathematical discourse that leads to opportunities for learning. We will describe a task that we have used successfully in an algebra course for K–12 in-service teachers as part of the Oregon Mathematics Leadership Institutes (OMLI). An analysis of both the task itself and the participants' mathematical activity as they engaged with the task will be provided with the goal of identifying and articulating:

1. the kinds of mathematical discourse that was generated;
2. the kinds of opportunities for mathematical learning that this discourse provided; and
3. design features of the task that seemed to support the discourse and subsequent learning opportunities.

The goal of this analysis is to generate design principles for developing tasks that promote rich mathematical discourse and to explicate the interaction between tasks and discourse in promoting learning.

## SETTING/METHODS

### *The Setting*

The OMLI algebra course consisted of 15 two-hour sessions over the course of three consecutive weeks. The section of the course from which we drew our data consisted of 16 students (usually seated in groups of four) and was co-facilitated by the first author and an experienced K–12 mathematics teacher and administrator with extensive experience facilitating professional development. Modes of instruction included work on tasks individually, in pairs, and in small groups as well as discussion in pairs, small groups, and whole class.

### *Methods*

Each session of the course was videotaped using two cameras to capture work in two of the four small groups as well as the activity at the front of the room and in the audience during whole class discussions. All of the students' written work was digitally copied and all of the posters created during class were digitally photographed. Additional data sources included instructional prompts, facilitation protocols, and notes regarding instructional design decisions.

The data corpus was analyzed using an iterative method derived from techniques described by Cobb and Whitenack (1996) and Lesh and Leh-

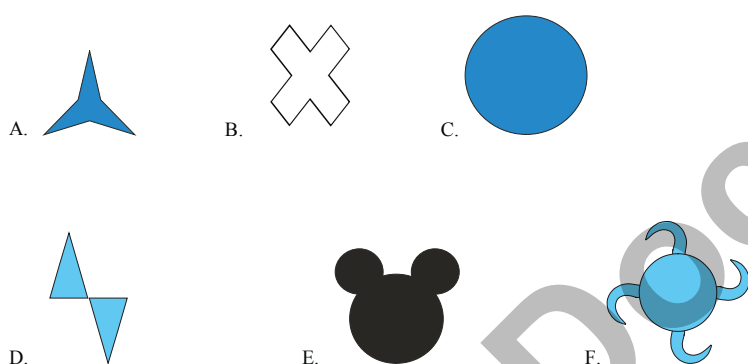
rer (2000). The first phase of our analysis consisted of viewing videotapes of the classroom activity to code the students' discourse using the OMLI Classroom Observation Protocol (Weaver, Dick, & Rigelman, 2005). The second phase of our analysis involved looking more closely at the justifications identified during the first phase of analysis. We analyzed the level of sophistication of the students' justifications and the nature of the bases (or foundations) of these justifications. We also attended to how these evolved throughout the task sequence. The third phase of our analysis focused on the mathematical content of the students' discourse. Specifically, we attended to opportunities for mathematical learning that resulted from the mathematical discourse, paying special attention to notions of equivalence as that was an important learning goal for the sequence. The fourth phase of our analysis focused on the task sequence itself in light of the earlier phases of analysis. Specifically, we worked to identify characteristics of the task sequence that seemed to promote the phenomena we had identified in our earlier analyses of the students' discourse.

### *The Measuring Symmetry Task Sequence*

In this section we provide the context for the analyses that follow by describing the first task of the Measuring Symmetry sequence, briefly describing the students' mathematical activity in response to this task, and describing a follow-up task that is the focus of much of the discourse we will consider. The Measuring Symmetry task sequence was the second instructional sequence of the course and was initiated on the third day of instruction. The starting point of the sequence was a task (Figure 1) that required the students to first rank six figures from least symmetric to most symmetric and then create a system for measuring (assigning a number to) the symmetry of any figure. The students were asked to do this by relying on their intuition and aesthetic sense.

This task seemed to generate a tremendous amount of substantive mathematical discourse among the students. After having approximately ten minutes to work individually on the first prompt (ranking the figures), the students took turns explaining their thinking to the other members of their small groups. These initial explanations revealed similarities and differences in approaches that were reflective of the different access points afforded by the task (e.g. some focused exclusively on bilateral symmetry while others included rotational symmetry in their considerations). Some students anticipated the second prompt and ranked the figures according to the number of symmetries (reflection and/or rotational) they had. Others used more informal ranking systems. Additionally, individuals varied according to whether they made an effort to break ties and in how they did

1. Order the figures below from least symmetry to most symmetry.



2. Create and describe in detail a method for measuring a figure's symmetry.

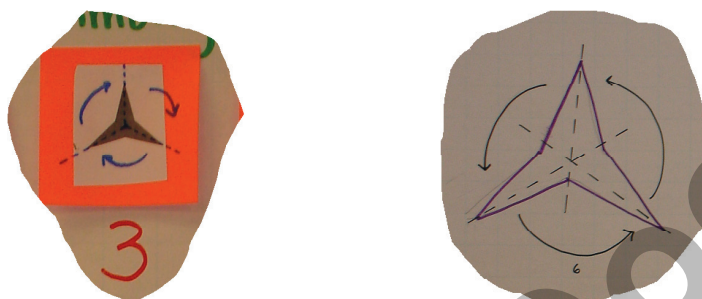
**FIGURE 1. The Measuring Symmetry task.**

so. When the second prompt was revealed and the students worked to come to consensus on a system for measuring a figure's symmetry, they began to engage in more substantive discourse including generalizing and justifying. Much of the justification activity was focused on determining how to handle rotation symmetries. Various students offered up and justified approaches with arguments based on their intuition and aesthetic sense (which is consistent with the instructions given in the task). In this way, this initial task provided a safe context for mathematical argumentation (since these arguments were based on intuition other students could disagree with them without undermining their confidence).

The students' discourse around this initial task touched on a number of significant mathematical ideas with which we hoped to engage the students. For instance, the initial discourse raised the question of whether the notion of symmetry goes beyond the idea of bilateral symmetry. It also set the foundation for thinking of symmetries as objects (because the students were essentially counting them). Finally, and most relevant for the discussion that follows, the students' work set the foundation for exploring the notion of equivalence in general (and equivalence of symmetries in particular).

Much of the discourse we will consider in the next two sections was focused on a follow-up task that we designed for the second day of the measuring symmetry sequence. During the first day of the sequence, each group of students developed a poster illustrating their ranking of the fig-

Consider the following poster scraps. How would you respond to the following question: “Each of these say that the figure has rotational symmetries, but aren’t there really 6 different rotational symmetries depicted?”



**FIGURE 2.** The follow-up question created from student answers to the original task.

ures and their method of quantifying a figure’s symmetry. We took digital photographs of the posters and the next day posed a question (Figure 2) that compared aspects of two different systems (one that counted counter-clockwise rotations and one that counted clockwise rotations). We asked whether the depicted figure had three rotational symmetries as shown on each poster, or if it in fact had six rotational symmetries (three in each direction).

We will provide a detailed description and analysis of the students’ mathematical discourse around the Measuring Symmetry task and this follow-up task. This will feature an analysis of a whole class discussion that featured diverse and increasingly sophisticated mathematical discourse. We follow this with an analysis of the mathematical content of the students’ discourse, explicating the opportunities for learning about equivalence that resulted from the discourse. Finally, we examine the Measuring Symmetry sequence itself in an effort to identify characteristics that seemed to promote the discourse that we observed and the resulting opportunities for learning.

## RESULTS OF OUR ANALYSES

### *The Students Mathematical Discourse*

The first phase of our analysis consisted of viewing videotapes of the classroom activity to code the students’ discourse using the OMLI Classroom Observation Protocol (Weaver et al., 2005). This analysis revealed that the students engaged in a wide variety of types of discourse. Further analysis revealed that the sophistication of the justifications increased throughout the

engagement with the task. In this section we will elaborate the varied types and levels of discourse the students engaged in while working on the task.

### *The Original Task*

Since the first prompt asks the students to make a decision based on their intuition, the mathematical discourse began with students sharing their rankings. The students often explained their rationale for their rankings while describing them. Since the students used different systems for ranking the images, the early discourse also involved questioning from group members who were trying to understand their peers' reasoning. This was usually followed by a response, which often included an explanation or justification. The following group discussion excerpt illustrates this type of interaction.

Ruth was the first person in her group to share her ranking, which she constructed considering only reflection symmetry. Daniel shared next, and he included both rotational and reflection symmetry. The following excerpt picks up with the discussions of Daniel's ranking.

Daniel: Also if you fold it in half—if you fold it this way, and you fold it this way, and you can rotate this one...I used rotational symmetry too. So three lines of symmetry plus three points of rotational symmetry.

Ruth: Explain that one to me.

Daniel: If you take this point, and rotate around it, it would be the same picture.

Ruth: By changing what you are doing.

Amber: I put my pencil down, and I turn it, I get here, and I get here again, so it's the exact same figure. You probably can't turn Mickey so he lands back on.

Daniel: You can turn him all the way around, so it has at least one.

This discussion began with Daniel sharing his ranking, which included an explanation of how he chose his ranking. His explanation included ideas with which Ruth was unfamiliar, so she asked questions to understand Daniel's reasoning. Both Amber and Daniel offered answers to Ruth's questions by explaining their understanding of rotational symmetry. Daniel then challenged Amber's claim that "Mickey" does not have any rotational symmetry, and justified his challenge by arguing that this figure has a 360-degree (or full turn) rotational symmetry. By basing the discussion on their personal ideas about symmetry, the students were able to comfortably share their ideas as well as explain and justify them.

As the group's engagement with the task progressed, and they worked to come to a consensus, the number of justifications and challenges increased.



For instance, while the students tried to decide on the order of the images, they began to challenge each other's rankings. Sometimes the challenges included a justification for the challenge, and sometimes justifications were offered as students rebutted the challenges. In the example that follows both types of justifications occur.

Amber: We are going with Mickey.

Ruth: Because of the line of symmetry.

Amber: And we are giving the line of symmetry a higher priority.

Daniel: But he only has one line of symmetry and this one has three rotations.

Amber: I know, but we are still going to give it a higher priority, because people see it better. So we have 3 lines, 2 lines, 1 line, and then no lines.

At this point the students were beginning to engage more in argumentation than in sharing their opinions. However, their justifications were still largely based on their opinions about symmetry.

In response to the second prompt the students began to generalize beyond specific examples. In this way they added another type of discourse to their discussions, which caused the discussions to increase in mathematical sophistication. The following excerpt illustrates this shift to thinking in general terms rather than about the particular images from the task.

Abby: What about if we give an aesthetic scale, it is it a 1, 2, or 3?

Amber: What if we had one figure in one room and another figure in another room, am I able to compare them, am I able to assign an aesthetic value to one of those figures? I don't know. I don't think so. I'm able to measure the symmetries and give it a numerical value, but aesthetic is hard.

Abby: I understand what you are saying, but that is not how we set up this table, we said Mickey Mouse is fun. That's not quantified yet.

Amber: But we added a quantifiable reason for it, because it has lines and not rotations.

In this discussion the students moved from talking about how symmetric these shapes were, to what made a figure more symmetric in general (which was necessary in order to develop a method for quantifying the symmetry of a figure). Note that their initial consideration of generalization comes in the form of challenges and justifications. Therefore, through the process of challenging and justifying the students began to generalize.



***The Follow-Up Task***

A similar diversity of types of discourse arose during the follow-up task. The discussions often took a similar form to those around the first task, but with more of an emphasis on explaining, challenging, and justifying. One example of this pattern can be seen in the following whole-class discussion excerpt.

Penny: You're rotating clockwise in the first figure. And you're rotating counterclockwise in the second figure. So you have a positive angle of rotation in the first one and a negative angle of rotation in the second one.

Susan: So what would you call that?

Penny: I would call it six.

Susan: You would call it six?

Penny: I would call it six.

Kathy: Well I count them differently. I don't count them as the number of turns, I think of them as degrees also. But for example on the first one, if you do that twice—

Instructor: Which one?

Kathy: The one on the left, if you turn that one twice and I numbered them I put point 1, point 2, point 3. Then where point one is located and you take the one on the right and you only turn it once to the left, those two pictures are exactly the same. So moving this one twice, the one on the left, twice and this one back once, the orientation of the picture is exactly the same. So those two rotations basically are the same thing. Since the position of the points are in the same location and you've double counted. You're counting positions twice.

Susan: So it's kind of like whether you're counting the motion as the thing or the ending position as the thing.

Kathy: Right and I'm saying they overlap each other.

In this excerpt both Penny and Kathy share their opinions about the question, and they accompany their answers with justifications. Kathy's answer, and accompanying justification, was presented as a challenge to Penny's response. Susan asked questions to help clarify her understanding of both arguments.

The discussion of whether the motions are important (or only the results) morphed into a discussion of different types of equivalence. This conversation also featured a diversity of types of discourse, but justification began to be built on earlier class discussions and other mathematical ideas rather than on opinions.

Sophie: 24, The number 24, you can say  $2 \times 12$  it would get 24 and you can say  $4 \times 6$  and  $2(3) \times 8$  and they all come up with the same answer. And we're all comfortable with the fact that there's several different ways, several different paths, to get the same answer. So ending result is the same but we count all these problems kind of differently. And the sense that like we were saying negative and positive turns right, in the end result is the same but you took a different path and is that ok?

Jim: But if we tie back into our original definition that we've agreed on, then it's just a movement that changes a shape. Then if you're saying the movements OK then well I'm gonna move, I mean if you look at this (holds up two square post-it notes) ok we'll limit it to this this this or this this this (Shows turns in each direction), well here's a movement that changed it and now it's back to its (moves the figure in a large motion lifting one square a foot or so above the other and wiggling it around before bringing it back down to rest on the other square) well watch this one (Does even more complicated motion) (Laughter) and it gets into the infinite, I don't want infinite symmetry.

Instructor: That's a great point. That's why we have to decide, that some of these things are the same as other ones. So we need to pin down what do we mean by equivalent.

Amber: Well I think Sophie's example doesn't fit my idea of equivalence as much as saying  $2 \times 4$  and  $4 \times 2$ .

Sophie begins this discussion by sharing her opinion and relating it to something familiar to everyone in the classroom. Jim then challenges Sophie's idea, and justifies his challenge by relating his response to earlier class discussions and by modeling an illustrative example of his idea for his peers. Amber also challenges Sophie's statement, and justifies her opinion by using a mathematical example that is related to Sophie's.

### *Bases of Justifications*

Although the students engaged in justifications throughout the entirety of the task, the justifications changed form throughout the task because the students began to base their justifications on previous discussions. In this section, we explicate the different bases the students used as they progressed through the task.

In the initial part of the task students based their decisions on their opinions and intuition. This helped students feel confident in their answers, and enabled them to justify their answers. This meant, however, that the justifications were based entirely on the students' beliefs or opinions. An example of this is when Amber said "I know, but we are still going to give it

a higher priority, because people see it better.” In this case she was basing her argument on her beliefs about people’s views of line symmetry versus rotational symmetry.

As the small groups came to a consensus on their rankings, they were then able to use those agreements as bases for their justifications. When Abby said, “I understand what you are saying, but that is not how we set up this table, we said Mickey mouse is fun. That’s not quantified yet,” she was basing her justification on the consensus her group had reached earlier. In this way her justification was no longer based on opinions or beliefs, but on something that had been established by the group.

Once again, the agreements reached from the argumentation described above moved on to become the bases of future justifications. Eventually the students created working definitions upon which they could base their justifications. This can be seen when Jim said “but if we tie back into our original definition that we’ve agreed on, then it’s just a movement that changes a shape.” In this way, Jim used the class’s emerging definition of symmetry to justify his view of equivalent symmetries.

The students’ justifications continued to be based on increasingly sophisticated ideas. For instance, as the class worked on defining equivalent symmetries, Kathy suggested that two motions could be considered equivalent if all of the points of a figure end in the same location after each motion is applied to the figure. The students were then able to use Kathy’s definition to justify their answers to later questions. Jim did this when he argued that no reflection could be equivalent to a rotation by saying “No, because one of the points is still in the same spot, but the other two have shifted.” In this way the students began to engage in more sophisticated and rigorous justification as they began to base their justifications on emerging mathematical definitions.

Above we saw that the variety of types of discourse the students produced expanded throughout their engagement with the task. Here we saw that not only the variety of types of discourse increased, but the sophistication increased as well. That is, the students went from basing their arguments on their opinions to basing their arguments on more mathematical foundations.

#### *Opportunities to Learn about Equivalence that Resulted from the Students’ Discourse*

Equivalence is one of the most important ideas at all levels and in all areas of mathematics. Equivalence was a central theme of the OMLI Algebraic Structures course and we explored notions of equivalence that spanned the K-16 grade range. At various times we considered equivalent arithmetic

expressions (e.g.  $2 + 2$  and  $1 + 1 + 1 + 1$ ), equivalent algebraic expressions (e.g.  $a(b + c)$  and  $ab + ac$ ), and equivalent algebraic structures (e.g. group isomorphism). The notion of equivalent symmetries is somewhat paradigmatic in the sense that it involves focusing on the sameness of the result of a process rather than the process itself. This idea underlies many examples of equivalence (including equivalent arithmetic and algebraic expressions) and is captured formally in the definition of equivalence of functions. Two functions are equivalent if they always produce the same output given the same input, and of course equivalence of symmetry transformations is a special case of equivalence of functions. We designed the Measuring Symmetry sequence in part to support our overarching goal of enriching the students' conceptions of equivalence. In this section, we take another look at the whole class discussions analyzed in the previous section, this time attending to the opportunities for learning about equivalence that were afforded by the students' discourse.

*Tension Between Attending to Sameness and Attending to Differences*

The first phase of the whole class discussion featured a back and forth between students arguing for the differences between various symmetries and students arguing for the sameness of various symmetries. Penny led off with an argument that clockwise rotations are different than counterclockwise rotations. Penny's argument was based on the fact that the motions felt different to her.

Penny: You're rotating clockwise in the first figure. And you're rotating counterclockwise in the second figure. So you have a positive angle of rotation in the first one and a negative angle of rotation in the second one.

Susan: So what would you call it?

Penny: I would call it six

Kathy then responded by arguing that a 240-degree clockwise rotation and a 120-degree counterclockwise rotation are the same because they change the orientation of the figure in the same way. Kathy's argument was based on the fact that she felt that it was the ending position that matters and not the motion.

Kathy: Well I count them differently. I don't count them as the number of turns, I think of them as degrees also. But for example on the first one, if you do that twice —

Instructor: Which one?

Kathy: The one on the left, if you turn that one twice and I numbered them I put point 1, point 2, point 3. Then where point one is lo-

cated and you take the one on the right and you only turn it once to the left, those two pictures are the exactly the same. So moving this one twice, the one on the left, twice and this one back once, the orientation of the picture is exactly the same. So those two rotations basically are the same thing. Since the position of the points are in the same location you've double counted. You're counting positions twice.

Susan: So it's kind of like whether you're counting the motion as the thing or the ending position as the thing.

Note that Susan made an important contribution by pointing out that Kathy's approach (which argues for a weaker condition for equivalence) involved attending to the effect the motion has on the figure rather than merely the motion itself, whereas Penny's approach (which argues for a very strict condition for equivalence) involves attending almost exclusively to the actual motion. As a result the debate shifted slightly, but significantly, to one in which arguments were given either in support of or against the idea of focusing on the effect of the motion. This allowed the students to shift from focusing on their beliefs to relating their arguments to other mathematical ideas.

The discussion began with Sophie arguing that in arithmetic different multiplication problems are considered to be different even if they produce the same answers.

Sophie: 24, The number 24, you can say  $2 \times 12$  it would get 24 and you can say  $4 \times 6$  and  $2(3) \times 8$  and they all come up with the same answer. And we're all comfortable with the fact that there's several different ways, several different paths, to get the same answer. So ending result is the same but we count all these problems kind of differently. And the sense that like we were saying negative and positive turns right, if the end result is the same but you took a different path and is that ok?

Jim then brought the discussion back to the original task (ranking/measuring symmetries) and to the class's definition of symmetry, arguing that if one focuses only on the motion when determining equivalence then all of the figures have infinitely many symmetries (which would of course render the measurement system meaningless).

Jim: But if we tie back into our original definition that we've agreed on, then it's just a movement that changes a shape. Then if you're saying the movements ok then well I'm gonna move, I mean if you look at this (holds up two square post-it notes) ok we'll limit it to

this this this or this this this (shows turns in each direction), well here's a movement that changed it and now it's back to its (moves the figure in a large motion lifting one square a foot or so above the other and wiggling it around before bringing it back down to rest on the other square) well watch this one (does even more complicated motion) (laughter) and it gets into the infinite, I don't want infinite symmetry.

#### *Articulating and Arguing with an Equivalence Criterion*

Following the whole-class discussion, the students were asked to develop a system for discerning whether two symmetries were equivalent. Kathy presented her approach to the class.

Kathy: My triangles have the different numbers. So here is my starting one with the 1, 3, and 2. And if I rotate it a 120 my 3, 2, and 1 move and if I rotate 240, they move again. So a 120 and a 240 and a 360 are all three different orientations. But 360 and 720 they're the same orientation. So it takes care of Jim's problem where 360 and 720 and everything, they're all the same. If you go with going the other direction and you go a negative 120. Notice a negative 120 and a 240 have the same orientation so they're considered the same symmetry. But the 240 and the 120 are different because the numbers are in a different location. Does that help?

Then one of the instructors asked the class whether they could use Kathy's idea to determine whether a reflection (a flip across the vertical in particular) could be equivalent to a rotation. Immediately Jim and Susan (among others) said this reflection could not be equivalent to a rotation. Jim quickly gave a justification using Kathy's working definition of equivalence, arguing that a reflection cannot be equivalent to a rotation because the ending orientation of the figure will necessarily change in a different way than it would under a rotation.

Jim: No, because one of the points is still in the same spot, but the other two have shifted.

In this discussion Jim (and other students) based their argument on the emerging definition of equivalence (based on Kathy's idea) rather than on their own opinions about equivalence. In this process the class began to make sense of Kathy's idea, which helped them begin to evaluate it. This was an important part of the process of creating a class definition of equivalence and the students' learning of this concept.

**Summary**

As they engaged with the measuring symmetry task (and particularly the follow-up question), the students engaged in substantive discourse around the issue of equivalence. By beginning with their intuitive notions about equivalence and extending to the class' working definitions, this discourse led to a tension between attending to similarities among symmetries (in particular the impact they have on the figure) and attending to differences between symmetries (in particular the differences between the actual motions performed). This tension created an intellectual need to develop a notion of equivalence that could resolve the tension in a way that was consistent with the students' informal notions of symmetry as seen in their initial measurement schemes. To do this the students needed to stop relying exclusively on their intuition about the task, and to base their arguments more on class consensuses and working definitions, which had evolved from their intuition and mathematical explorations. Thus the task both created a need to consider the issue of equivalence and a context for supporting mathematically meaningful arguments for and against proposed notions of equivalence (an hence support for formulating a definition of equivalence.)

*Characteristics of the Measuring Symmetry Task Sequence*

In this section, we will analyze some aspects of the measuring symmetry sequence with an eye toward teasing out those characteristics that supported the productive mathematical discourse we observed and the learning opportunities that resulted from that discourse. The characteristics described here emerged as a result of coordinating our analysis of the students' discourse with our analysis of the resulting opportunities for learning. These analyses were significantly influenced by our awareness and experience with several frameworks related to instructional design. In particular, we will draw connections between our observations and the models and modeling perspective (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003), Harel's (2007) DNR framework, and Rasmussen and Marrongelle's (2006) notion of a pedagogical content tool.<sup>2</sup> Relevant aspects of these frameworks will be elaborated in the context of explicating our observations.

In the following subsections, we elaborate four characteristics of the measuring symmetry task that seem to have contributed to what we observed in the students' mathematical discourse:

1. The task sequence provided multiple access points for the students.

<sup>2</sup>The more general instructional design theory of Realistic Mathematics Education (Gravemeijer, 1999) also influences our analysis and is related in many ways to the more specific constructs we describe.



2. The task called for the creation of a mathematically significant model.
3. The task (and the follow-up question) admitted multiple *justifiable* approaches.
4. The task provided a context that could be leveraged to bring the argumentation to a resolution.

#### *The Tasks Provided Multiple Access Points*

The initial measuring symmetry task was one that was accessible to all of the students. All of the students had some aesthetic sense of symmetry that they relied on as they formulated their responses, while some of the students could approach the task more formally based on their experiences. For example some of the elementary teachers mentioned having taught bilateral symmetry to their students. However, even without those kinds of experiences, students could make progress on the task by analyzing the figures in an effort to quantify their intuitive sense of symmetry. Regardless of how the students approached the problem they had an answer to share and explanations for their answers.

This characteristic of the task is in line with the *reality principle* from the models and modeling perspective (Lesh et al., 2003). The reality principle is one of the principles for designing what Lesh et al. refer to as model-eliciting activities. The reality principle asks the question: Will students make sense of the situation by extending their own knowledge and experiences? In the Measuring Symmetry sequence the students relied on their own aesthetic sense and experiences (formal or otherwise) with symmetry in order to produce their rankings and measurement systems. The task had the characteristic that it could be approached using whatever prior knowledge or experiences the individual students brought to the task. And the variety of responses that resulted provided a rich context for the students' discourse. That is, it provided each student an entry point to the discussion.

#### *The Tasks Called for the Creation of a Mathematically-Significant Model*

Another of the principles for the design of model eliciting activities described by Lesh et al. (2003) is the *model construction principle*. An activity satisfies this principle if it immerses the students in a situation in which they are likely to confront the need to develop (or refine, or modify, or extend) a mathematically-significant construct. In the case of the measuring symmetry sequence, the students are asked to develop a method for quantify-

ing the symmetry of a figure. In order to create such a system, the students needed to consider a number of questions either explicitly or implicitly:

- Do rotational symmetries count?
- If rotational symmetries count, should they be weighted less than reflection symmetries?
- Does a 360 degree or 0 degree rotation count?
- When should two symmetries be considered to be the same?

In order to answer these questions and produce their final quantification scheme, the students needed to debate with each other. This means that students needed to ask questions to understand each other's ideas, explain their ideas, challenge their ideas, and justify their ideas. In this way, this characteristic of the task seems to be largely responsible for the patterns of discourse and opportunities for learning that we observed.

Additionally, the model construction principle requires that the construct the students are asked to produce be mathematically significant. In the case of the measuring symmetry task, the model the students are asked to produce must account for a number of significant issues. In particular, a method for quantifying the symmetry will depend on a method for determining whether two symmetries are equivalent. In this sense the task creates what Harel (2007) refers to as an *intellectual need* to determine what it means for two figures to be equivalent. The follow-up task served to make this need more pressing by confronting the students with a discrepancy in how two groups were counting symmetries that could only be resolved by developing a notion of equivalence. So the measuring symmetry sequence calls for the creation of the very mathematically significant construct of equivalence of symmetries. This pressed the students to base their justifications on increasingly significant ideas. The combination of these characteristics seemed to contribute significantly to the opportunities for learning that we observed.

#### ***The Tasks Admitted Multiple Justifiable Approaches***

The initial tasks in the Measuring Symmetry sequence could be approached in any number of different and justifiable ways. For instance when ranking figures, it is reasonable to include a 360-degree rotation as a symmetry (which some groups did). However, it is also reasonable to exclude a 360-degree rotation. Furthermore, these different approaches are mathematically justifiable. For instance, students argued that a 360-degree rotation is much like a zero in addition (in that it does not change the figure) and so this rotation should be included. On the other hand, students argued that every figure has a 360-degree rotation and so including this rotation artificially inflates the symmetry measurement of every figure and

as a result the measurement method does not accurately capture the idea of having no symmetry.

The follow-up task also admitted more than one justifiable response. As we saw above, students were able to provide convincing arguments that one should not consider a 240-degree clockwise rotation to be the same as a 120-degree counterclockwise rotation, while others were able to argue convincingly that one should indeed consider these to be the same. Clearly this characteristic of the task (and the follow-up) helped to set the stage for the mathematical argumentation that we observed by allowing for two opposing positions to be considered and justified.

In this way, this follow-up task seems related to what Rasmussen and Marongelle (2006) call a *generative alternative*. “Generative alternatives are defined as alternate symbolic expressions or graphical representations that a teacher uses to foster particular social norms for explanation and that generate student justifications for the validity of these alternatives.” (p. 389). In this case, we used the two ways of thinking about rotations (clockwise and counterclockwise) to pose two alternative ways to count these (as three rotations or as six rotations). And as we have shown above, the students did indeed provide justifications for each of these alternatives. And these justifications were crucial to the eventual development of the notion of equivalence of symmetries.

#### ***The Tasks Provided a Context that Supported the Resolution of the Argumentation***

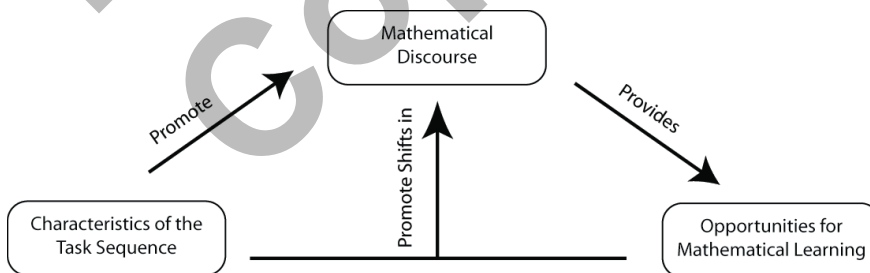
While we certainly see the intrinsic value of having students engage in mathematical discourse, we also wish to emphasize that an important desired outcome of mathematical discourse is the learning of specific mathematical content. The first three characteristics of the measuring symmetry sequence that we have elaborated all seem to have contributed to the generation of significant mathematical discourse on the part of the students, setting the stage for learning about equivalence. However the task had another feature that seemed to be particularly important for moving the students’ thinking forward. The task provided a context for resolving the mathematical arguments that it generated. In this case, the students needed to make a decision to consider different symmetries to be equivalent if they were to develop a system for measuring symmetry that was mathematically meaningful. Additionally, this decision needed to be based on earlier arguments and decisions. We saw this in Jim’s argument that closed the whole class discussion by appealing to the emerging definition of symmetry and the goal of meaningfully measuring symmetry. This characteristic is consistent with the *self-evaluation* principle described by Lesh et al. (2003), in which an activity should promote self-evaluation on the part of the students. The measuring symmetry context provided criteria for the students

to evaluate their stance on whether to focus on the similarities or differences between symmetries. This feature cemented the intellectual need to develop a notion of equivalence that focused on the result of the symmetry and not on the actual motion performed.

### CONCLUSIONS

In this paper we presented an analysis of a group of students' (teachers in this case) discourse as they engaged with the Measuring Symmetry sequence and examined the opportunities for learning that participating in this discourse afforded them. We also examined the task sequence itself and identified some characteristics that seemed to promote the kinds of mathematical activity we observed. These analyses point to an important interaction between the characteristics of the task sequence, the discourse in which the students engaged, and the opportunities for learning that resulted. This interaction is illustrated in Figure 3.

Our analysis suggests that specific aspects of the task promoted the kinds of discourse we saw as the students engaged with the initial tasks. For example, the students could approach the task by relying on their own intuition and aesthetic sense. This seemed to be a basis from which they felt comfortable providing justifications. Then, as the students shared their perspectives, they were exposed to other ways of thinking about symmetry (for example ways that included attention to rotational symmetry). This provided a chance for individual students to develop more expansive notions of symmetry which in turn set the stage for more sophisticated levels of discourse (for example the discussions about whether some rotations should be considered equivalent to each other.) Aspects of the follow-up task further promoted this shift by presenting students with two options for counting symmetries, resulting in mathematical argumentation to resolve



**FIGURE 3. Illustration of the interaction between tasks, mathematical discourse, and opportunities for mathematical learning.**

the question of whether (and, if so, under what conditions) two different motions should be considered to be equivalent symmetries. Ultimately this discourse supported the students' development of the concept of equivalence, which was an important mathematical goal of the course.

To summarize, we found that four characteristics of the Measuring Symmetry task sequence were particularly important in terms of promoting substantive mathematical discourse and as a result providing opportunities for mathematical learning:

- The tasks provided multiple access points.
- The tasks called for the creation of a mathematically significant model.
- The tasks admitted multiple *justifiable* approaches.
- The tasks provided a context that supported the resolution of the argumentation.

The first characteristic allowed all of the students to contribute to the mathematical discourse and engage with the task. The second characteristic ensured that the discourse would focus on mathematically significant ideas. The third characteristic set the stage for a substantive mathematical debate. Finally, the fourth characteristic supported the instructional goal (in this case the learning of equivalence) by providing a means to evaluate various approaches on the way to settling on the most mathematically viable approach.

Together these aspects of the Measuring Symmetry task seemed to support rich mathematical discourse and as a result provided opportunities for learning. Further research is needed to determine the extent to which these characteristics (or others) are necessary and/or sufficient. Nonetheless, this case of the Measuring Symmetry task sequence illustrates the important role that task design can play in promoting discourse and learning, and begins the process of identifying design principles for effective tasks.

## REFERENCES

- Cobb, P., Boufi, A., McClain, K., & Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal for Research in Mathematics Education*, 28(3), 258–277.
- Cobb, P., & Whitenack, J. (1996). A method for conducting longitudinal analyses of classroom videorecordings and transcripts. *Educational Studies in Mathematics*, 30, 213–228.
- Forman, E. A. (2003). A sociocultural approach to mathematical reform: speaking, inscribing, and doing mathematics within communities of practice. In Kilpatrick, J., Martin, W. G., & Schifter, D. (Eds.), *A research companion to principles*

- and standards for school mathematics* (pp. 333–352). Reston, VA: National Council of Teachers of Mathematics.
- Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, 1(2), 155–177.
- Harel, G. (2007). The DNR system as a conceptual framework for curriculum development and instruction. In Lesh, R., Kaput, J., & Hamilton, E. (Eds.), *Foundations for the future in mathematics education*. Mahwah, NJ: Erlbaum.
- Hatano, G., & Inagaki, K. (1991). Sharing Cognition through collective comprehension activity. In Resnick, L. B., Levine, J. M., & Teasley, S. D. (Eds.), *Perspectives on socially shared cognition* (pp. 331–348). Washington, DC: American Psychological Association.
- Hiebert, J. (2003). What research says about the NCTM standards. In Kilpatrick, J., Martin, W. G., & Schifter, D. (Eds.), *A research companion to principles and standards for school mathematics* (pp. 5–23). Reston, VA: National Council of Teachers of Mathematics.
- Himmelberger, K. S., & Schwartz, D. L. (2007). It's a home run! Using mathematical discourse to support the learning of statistics. *Mathematics Teacher*, 101(4), 250–256.
- Ilaria, D. (2002). *Questions that engage students in mathematical thinking*. Paper presented at the 24th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics.
- Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *Elementary School Journal*, 102, 59–80.
- Kysh, J., Thompson, A., & Vicinus, P. (2007). Welcome to the MT 2007 focus issue: Mathematical discourse. *Mathematics Teacher*, 101(4), 245–246.
- Lampert, M. (1986). Teaching Multiplication. *Journal of Mathematical Behavior*, 5(3), 241–280.
- Lampert, M., & Cobb, P. (2003). Communication and language. In Kilpatrick, J., Martin, W. G., & Schifter, D. (Eds.), *A research companion to principles and standards for school mathematics* (pp. 237–249). Reston, VA: National Council of Teachers of Mathematics.
- Lesh, R., Cramer, K., Doerr, H., Post, T., & Zawojewski, J. (2003). Model development sequences. In Lesh, R., & Doerr, H. (Eds.), *Beyond constructivism* (pp. 35–58). Mahwah, NJ: Lawrence Erlbaum.
- Lesh, R., & Lehrer, R. (2000). Iterative refinement cycles for videotape analyses of conceptual change. In Kelly, A. & Lesh, R. (Eds.), *Handbook of research design in mathematics and science education* (pp. 665–708). Mahwah, NJ: Erlbaum.
- National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- Rasmussen, C., & Marrongelle, K. (2006). Pedagogical content tools: Integrating student reasoning and mathematics in instruction. *Journal for Research in Mathematics Education*, 37, 388–420.
- Simon, M., & Blume, G. W. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *Journal of Mathematical Behavior*, 15, 3–31.

Staples, M., & Colonis, M. M. (2007). Making the most of mathematical discussions. *Mathematics Teacher*, 101(4), 257–261.

Stein, C. C. (2007). Let's talk: Promoting mathematicalhWeaver, D., Dick, T., & Rigelman, N. (2005). *Assessing the quality and quantity of student discourse in mathematics classrooms*. Portland, OR: RMC Research Corporation.

**REFERENCE NOT CITED IN TEXT:**

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, 1(2), 155–177.