TOTOM 2010 **What's the deal with algebraic structure?**

The following activity and questions were used in conjunction with reading an article from *Mathematics Teaching in the Middle School*, "Using Clock Arithmetic to Teach Algebra Concepts" by Elizabeth M. Brown and Elizabeth Jones (September 2005). The model I used to develop this activity and the questions to accompany the reading was NCTM's Reflection Guides, focusing students' attention on the mathematical content rather than pedagogy. The primary mathematical idea was the necessity of field properties in solving linear equations.

Part 1

The students worked on the attached activity *Introduction to Clock Arithmetic* prior to reading the article. Some of the questions on the activity paralleled the questions considered by the students in the middle school classroom described in the article. I wanted the students to grapple with some of the mathematical concepts described in the article before reading it.

Part 2

The students read the article and answered the attached questions. The questions were written to encourage the students to unpack the mathematical ideas addressed in the article.

Part 3

The students' last assignment included a follow-up question to assess their understanding of the significance of field properties in solving linear equations.

Class activity **Introduction to clock arithmetic**

If we consider the 12 numbers on a clock face, we get an arithmetic on a finite set with some properties that are similar to arithmetic with real numbers, but other properties that are different.

Complete the following addition table for clock arithmetic. The symbol $\oplus$ is used for addition to distinguish this addition from addition defined for real numbers.

$\oplus$	1	2	3	4	5	6	7	8	9	10	11	12
1												
2												
3												
4												
5												
6												
7												
8												
9												
10												
11												
12												

- List any patterns you observe in the addition table. Did you use any of these patterns to help you fill in the table?

- What number plays the role of "0"?

- Each real number has an additive inverse, that is, for each element $\alpha \in \mathbb{R}$ there exists $-\alpha \in \mathbb{R}$ such that $\alpha + \left(-\alpha\right) = 0$. Does each number in the set of numbers that comprise clock addition have an additive inverse? If so, list the inverse of each number. Describe how one can identify additive inverses using the table.

Multiplication is defined as repeated addition. For example,

$3 \odot 5 = 5 \oplus 5 \oplus 5 = 3$.

Use this definition of multiplication to complete the multiplication table for clock arithmetic.

$\odot$	1	2	3	4	5	6	7	8	9	10	11	12
1												
2												
3												
4												
5												
6												
7												
8												
9												
10												
11												
12												

- List any patterns you observe in the multiplication table. Did you use any of these patterns to help you fill in the table?

A *group* is the simplest algebraic structure.

A group G is a set along with a rule (called a law of composition), which to each pair of elements $x, y \in G$ associates an element denoted by xy in G having the following properties.

- GR 1: For all $x, y, z \in G$, $(xy)z = x(yz)$
- GR 2: There exists an element e of G such that $ex = xe = x$ for all x in G.
- GR 3: If $x \in G$, then there exists an element $y \in G$ such that $xy = yx = e$.

The first group property says that the operation is associative, the second property says that there is an identity with respect to the defined operation and the third property says that each element has an inverse with respect to the operation.

- We defined two rules (operations) on the set of numbers on a clock, $\oplus$ and $\odot$. Is this set with either of these rules a group?

Questions to accompany the reading

- On p. 108 of the article *Using Clock Arithmetic to Teach Algebra Concepts* the authors describe using the multiplication table to solve division problems. They write, "We then give our students several division problems to solve, including those with multiple solutions or no solution and compare this to division by zero in the set of integers."

  Find an example of a division problem with multiple solutions.

  Find an example of a division problem with no solution.

  What number is like zero in clock arithmetic? Show that division by this number is undefined.

  The authors note, "[Solving division problems] not only reinforces the relationship between multiplication and division but also prepares students to solve equations of the form $7x = 4$." Above you found division problems with multiple solutions and division problems with no solution. How is this related to solving linear equations and/or dividing by zero?

- On p. 109 the authors write, "Questions such as 'Does $5 \oplus 9 = 5 \ominus 3$?' also reinforces the idea of additive inverses." Highlight the role of the additive inverse in this question.

- Solve the equation $6x \oplus 4 = 1$. The variable and constants are elements of the finite system $\mathbb{Z}_7$.

Follow-up questions on the last assignment

This week we solved an equation over the finite field $\mathbb{Z}_7$. Ann chose $\mathbb{Z}_7$ for this activity because $\mathbb{Z}_{12}$ with clock addition and multiplication is *not* a field.

a. What field property (or properties) does not hold for $\mathbb{Z}_{12}$ with clock addition and multiplication?

b. Why are *all* the field properties needed to solve any given linear equation? Illustrate your response by giving an example of a linear equation that can be solved over $\mathbb{Z}_{12}$ with clock addition and/or multiplication and a second example of an equation that cannot be solved.