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MULTICULTURAL MATHEMATICS AND ALTERNATIVE ALGORITHMS

Different cultures use different arithmetic procedures

Up until recently I wasn't even aware that other people in the world did things [arithmetic algorithms] differently. I thought God sent these. That's the way of the world. The first day you [to another teacher] were talking about some way you did things differently in Ireland. It never occurred to me. I thought there was a world standard.

—A sixth-grade teacher reflecting on alternative-mathematical algorithms

A teacher's beliefs about mathematics significantly affect the manner in which he or she teaches (Thompson 1992). Teachers, from school experience, often believe that there is one right way to solve a particular mathematics problem or to apply a computational algorithm for adding, subtracting, multiplying, and dividing. In turn, these beliefs become the beliefs of their students. The NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) has called for decreasing the attention paid to isolated treatment of paper-and-pencil computations and the memorization of rules and algorithms and suggests instead that we increase the attention paid to students' *creating* algorithms and procedures. Implicit in this suggestion is that the algorithms we have come to learn and to use are not the only way, and may not even be the best way, to compute.

Although teachers are usually aware that various cultures have historically used algorithms that are different from those currently taught in United States schools, these teachers may not be aware that various algorithms are being used currently in the United States. Many of these algorithms are culturally based and are used by people with common ethnic and cultural backgrounds. This article describes how preservice elementary school teachers developed an awareness that the algorithms we teach in school are not the only algorithms for operating on numbers and that if they look, they may find alternative algorithms in their community and school.

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An Invented Algorithm

Dictionaries define an algorithm as a rule or procedure for solving a problem. Computational algorithms are invented by people to streamline the process by which we compute. The fact that algorithms are a convention is often lost on our students, who come to think of a particular algorithm as *the* way, instead of as *a* way, to compute. The following example illustrates the role that algorithms play in school mathematics.

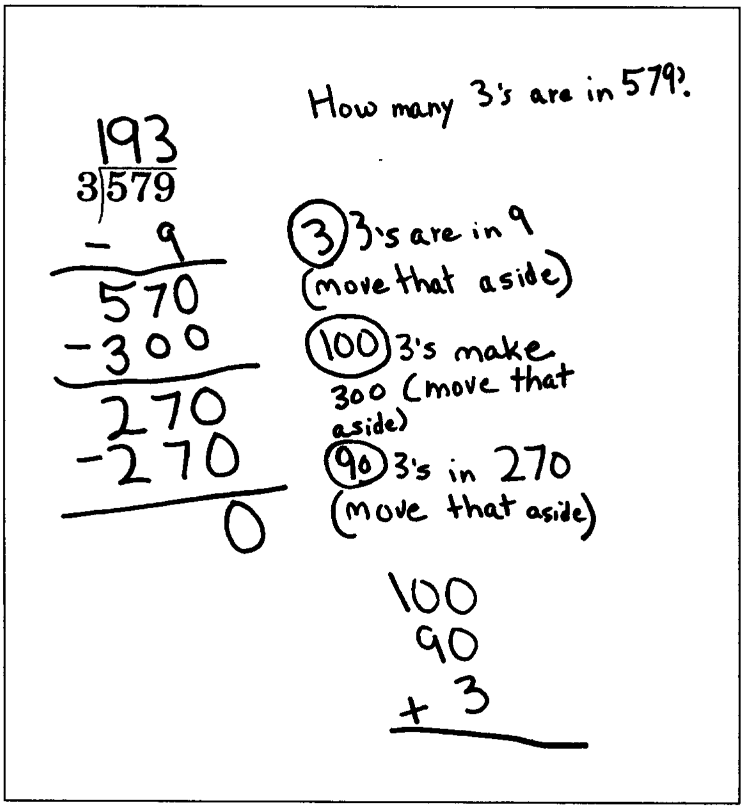
A colleague recently told me a story about his third-grade daughter, who came home from school crying because of long division. The girl, whom I shall call Michelle, could not understand why she needed to learn a procedure for 63 ÷ 7 or 88 ÷ 8. After all, she said, "Can't everyone see what the answers to those are?" Michelle was struggling with the procedure for long division taught in class and was getting confused about when to multiply, when to subtract, and when to "bring down the next number." That afternoon her father sat with her and took a fresh approach. He first asked her whether she could explain a way of thinking about 126 divided by 3. Michelle said, "If you share 126 with 3 people, how much would each person get?" Her dad then asked if she could think of another approach, and she said, "How many 3s are in 126?" He told her to think about division that way. He asked her to imagine having a large number of ones, to take out groups of three, and to keep track of how many groups she "moved aside." Michelle thought that the explanation made sense, and without any other prompting, she solved 579 ÷ 3 (see **fig. 1**).

Although Michelle's dad suggested that she write down only what she needed, Michelle said that it helped her to write "How many 3s are in 579?" so she could remember what she was doing. Notice the unconventional approach that Michelle invented for this problem. This "algorithm," although nonroutine, was based on Michelle's understanding of the meaning of division and her sense for numeration. It makes complete sense and involves a deeper mathematical understanding on Michelle's part than would have been necessary in memorizing the conventional algorithm. Whereas Michelle invoked good numeration sense when working her solution, her dad told me that she did not consider place value when working the traditional long-division algorithm. Instead, she treated all the numbers as ones—"How many 3s are in 5?"—instead of in 500, and so on.

Michelle was confused in learning the school's long-division algorithm, which is taught because it is an efficient method for dividing numbers. However, this algorithm, which is often taught as a set of steps by which one will arrive at the correct answer, is often taught instrumentally, that is, without understanding. Michelle, who understood what division meant, was able to invent her own algorithm for solving the division problem. Other examples appear in the literature of students' inventing algorithms for mathematics (Kamii, Lewis, and Livingston 1993), but this article takes a different approach to the role of algorithms. Instead of additional examples from individual students, it presents examples that people from diverse ethnic and cultural backgrounds learned in school.

FIGURE 1

Michelle's algorithm for 579 ÷ 3



Culturally Based Alternative Algorithms

Southern California schools in general, and San Diego Unified Schools in particular, comprise a diverse multiethnic, multilingual population. Although teachers are working hard to find ways to incorporate the knowledge of various cultures, it is unclear how this goal might be accomplished in the area of mathematics. To give elementary-mathematics-methods students an opportunity both to acknowledge the mathematical diversity of their students and to challenge the belief that "God sent these [algorithms]," I have devised the "alternative algorithm" assignment. The purposes for this assignment are the following:

1. To develop an appreciation for the fact that various cultures have developed alternative algorithms to those commonly used in the United States
2. To reinforce the view that the algorithms we have come to use are simply a matter of convention and should be seen as *a* way, not *the* way, to compute
3. To support the view that one can make sense of computational algorithms and in so doing, develop a deeper understanding of place value and the meanings of operations

Student teachers are asked to identify algorithms that are being used in the community but that differ from those taught in the United States. As student teachers locate various algorithms, they are expected to describe why the algorithms work. In so doing, they must think hard about the underlying mathematical ideas. Some student teachers have found alternative algorithms through students in their own classes, either directly from the students or by asking students to talk with members of the student's family. This system not only legitimizes the mathematics learning of either the child or a member of the child's family but also presents an opportunity to honor this learning in both the child's eyes and, depending on what is done with the information, in the eyes of all the students in the class. This article describes some of the alternative algorithms that have been located by student teachers.

Alternative Algorithms for Addition

The traditional addition algorithm taught in the United States involves writing numbers in columns then adding the columns, starting with the smallest place value and moving to the left. For example, to add 465 + 190 + 676, the algorithm works as follows:

$$\begin{array}{r} {}^{2}4{}^{1}6\,5 \\ 1\,9\,0 \\ +\,6\,7\,6 \\ \hline 1\,3\,3\,1 \end{array}$$

In this example, the superscript 1 represents one group of 10 ones, or 10, being "carried," and the superscript 2 represents two groups of 10 tens, or 200, being "carried." Various forms of keeping track of groups of numbers were found among individuals from the Philippines, Japan, Germany, and Ireland. One parent shared the following addition algorithm, demonstrated by adding 98, 24, 99, and 25, that he had learned when he was in second grade in the Philippines:

$$\begin{array}{r} -\,9\,8 \\ 2\,4\,- \\ -\,9\,9\,- \\ +\,2\,5 \\ \hline 2\,4\,6 \end{array}$$

This algorithm differs from the traditional algorithm used in the United States in two ways. First, a dash is used to notate a group of ten. For example, 8 plus 4 equals 12, but instead of remembering "12," one remembers only "2" and places a dash to indicate that one group of 10 has been reached. The second difference is that the number carried is not written but instead can be determined from the number of dashes. An example of one's thinking while using this algorithm follows:

> Eight and 4 is 12 ("-" for 10, leaving 2); 2 and 9 is 11 ("-" for 10, leaving 1); 1 and 5 is 6. So I carry two dashes, or 20. Two (tens) and 9 (tens) is 11 ("-" for 10); 1 and 2 is 3, and 9 is 12 ("-" for 10); 2 and 2 is 4. Carry two dashes, or 200. So the answer is 246.

A woman of Japanese descent was asked to add 87, 65, and 49. She shared the following algorithm, which she had learned in college in Japan. She referred to it as the "scratch technique."

$$\begin{array}{r} \not{8}_{0}7\phantom{_{0}} \\ 6\,\not{5}_{2} \\ +\,\not{4}_{0}\not{9}_{1} \\ \hline 2\,0\,1\phantom{_{0}} \end{array}$$

This "scratch method" is similar to the algorithm described in the "Philippine" algorithm. First, as in the "Philippine" algorithm, this algorithm keeps track of groups of tens by overstriking the appropriate digits. However, in addition to keeping track of the groups of tens, this algorithm also keeps track of the leftovers. For example, one might use the following thinking with this algorithm:

> Seven plus 5 is 12, which is 10 (strike through the 5) and 2 is left over (subscript 2); 2 and 9 is 11, which is 10 (strike through the 9) and 1 (subscript 1). So we have 1 left, and carry two groups of ten; 2 (tens) and 8 (tens) is 10 (strike through the 8) and 0 (subscript 0); 0 and 6 is 6, and 4 is 10 (strike through the 4) and 0 (subscript 0). So I have 0 tens, and I must carry two groups of 10 tens, or 200. The answer is 201.

The American division algorithm is different from others

Other addition algorithms differed from the traditional United States algorithm only with respect to where the carried digit was written. A young girl of Irish descent shared this algorithm:

$$\begin{array}{r} 1\,2\,3 \\ +\,3_{1}7_{1}8 \\ \hline 5\,0\,1 \end{array}$$

A twenty-year-old Mexican man explained an addition algorithm in which the numbers to be carried were placed to the side. He called this algorithm "llevamos uno," or "we carry one":

$$\begin{array}{rl} 1\,9\,4 & \\ +\,4\,9 & 1\,1 \\ \hline 2\,4\,3 & 1 \end{array}$$

An older man educated in Switzerland and a man schooled in Canada in the early 1970s both demonstrated that they had learned to add by starting from the left-most column. The man from Switzerland worked the following two problems:

$$\begin{array}{r} 5\,9 \\ +\,1\,6 \\ \hline 6\,0 \\ 1\,5 \\ \hline 7\,5 \end{array} \qquad\qquad \begin{array}{r} 4\,8\,1 \\ +\,9\,2\,6 \\ \hline 1\,3\,0\,0 \\ 1\,0\,0 \\ 7 \\ \hline 1\,4\,0\,7 \end{array}$$

This algorithm is the one that many elementary school children in the United States invent when encouraged to do their own thinking. That is, when asked to add multidigit numbers, most children will naturally begin adding the digits with the largest place value. This procedure is quite natural for adults as well. For example, if two friends emptied their wallets to pool their money, would they first count the $20 bills or the $1 bills?

Alternative Algorithms for Subtraction

The traditional algorithm for subtraction in the United States involves "borrowing," or regrouping, from the *minuend,* the quantity from which the subtrahend is subtracted. For example, 347 – 169 can be solved by an individual who subtracts beginning with the *ones* column and then works toward the *hundreds* column:

$$\begin{array}{r} {}^{2}3\,{}^{1}{}^{3}\not{4}\,{}^{1}7 \\ -\;1\;6\;9 \\ \hline 1\;7\;8 \end{array}$$

People from various countries subtract by using an algorithm different from that used in the United States. Instead of "borrowing," or regrouping, from the minuend, they use what might be referred to as an *equal-addition algorithm,* whereby an equal amount is added to both the minuend and the subtrahend. For example, the previous subtraction problem might be worked as follows:

$$\begin{array}{r} 3\,{}^{1}4\,{}^{1}7 \\ -\,{}^{1}1\,{}^{1}6\;9 \\ \hline 1\;7\;8 \end{array}$$

The following shows the thinking that might accompany this problem:

> First I want to subtract 9 from 7. I cannot do that, so I will add 10 to the 7, making it 17, and I will add 1 (ten) to the 6 (60) to make it 7 (70). Now I can subtract 9 from 17, and that equals 8. Next I want to subtract 7 (6 + 1) from 4. I cannot do that. So I will add 10 (actually, 10 tens) to the 4 to make it 14, and I will add 1 (100) to the 1 (100) in the subtrahend. Now I can subtract 7 (6 + 1) from 14, leaving 7. Finally I want to subtract 2 (1 + 1) from 3, leaving 1. So the answer is 178.

One way to understand why this technique works is to think of comparing the ages of a fifty-seven-year-old man and his twenty-nine-year-old daughter. In ten years, the father will be sixty-seven years old and his daughter will be thirty-nine years old, but the difference in their ages will not change. This algorithm is based on finding the difference between 57 and 29 by adding the same number to both the minuend and subtrahend.

$$\begin{array}{r} 5\,{}^{1}7 \\ -\,{}^{1}2\;9 \\ \hline 2\;8 \end{array}$$

This technique was used by adults from various countries who remembered having been taught this algorithm as children. During this assignment, it became clear that even many adults who had been educated in the United States had learned algorithms that were different from those commonly taught in today's American schools. These adults came from Persia; Panama; Croatia; Germany; Ireland; Riverside, California; and Brooklyn, New York.

Earlier in this century, a discussion involved which method for subtraction ought to be taught in American schools. This controversy was laid to rest after Brownell (1947) and Brownell and Moser (1949) summarized the research evidence, concluding that an interaction existed between the type of subtraction method and the style of teaching. When subtraction was taught procedurally, the second approach, known as "augmenting," or equal addition, was easier to learn. When subtraction was taught conceptually, regrouping, or "borrowing," was easier to learn. This research was instrumental in determining the subtraction algorithm taught in schools.

A man recalled learning the following algorithm while attending primary school in central Italy. When learning addition and subtraction, he was not permitted to write any other numbers on his paper. As a result, he had to perform the computations in his head. To solve 375 – 137, he first mentally subtracted 300 – 100, resulting in 200. Then he mentally subtracted 270 – 30, resulting in 240. Finally he subtracted 245 – 7, which is 238. This process works from left to right and requires one to keep track of place value.

Alternative Algorithms for Multiplication

A ninety-six-year-old German woman, recalling her Russian father's approach to multiplying, demonstrated the following example by multiplying 230 by 17:

~~230~~	~~17~~
115	34
57	68
~~28~~	~~136~~
~~14~~	~~272~~
7	544
3	1088
1	2176
	3910

Starting with the first number in the left-hand column, one keeps dividing each number by 2 while multiplying the corresponding number in the right-hand column by 2. When the left-hand column's number is odd, divide by 2 and drop the remainder. This process continues until a 1 is obtained in the left-hand column. Draw a line through all even numbers in the left-hand column, along with their corresponding numbers in the right-hand column, and then add the right-hand column's numbers that have not been crossed out. The sum of the uncrossed-off numbers in the right-hand column is the product of 230 × 17. This algorithm works because whenever an even number appears in the left-hand column, dividing it by 2 and multiplying the corresponding right-hand-column number by 2 conserves the product. Conversely, whenever an odd number appears in the left-hand column, dividing it by 2 and dropping the remainder does not conserve the product. To compensate for the dropped remainder, one must add the number in the right-hand column. This algorithm is referred to in the literature as the *Russian peasant algorithm* and makes it possible to multiply whole numbers by knowing only how to halve and double numbers and add.

An American man of Mexican descent presented the following algorithm for multiplying numbers using the example of 46 × 37. He does all the calculations in his head:

$$\begin{array}{lr} 46 & 50-4 \\ \underline{\times 37} & \underline{40-3} \\ & -150+12 \\ +2000-160 & \\ \hline 2000-310+12 & =1702 \end{array}$$

This approach is a direct application of the distributive property and works for exactly the same reasons but in a slightly different way than the traditional multiplication algorithm.

Each of two men, one educated in South Africa and one in Belize, shared an algorithm that started by multiplying the larger digits. Specific examples of how they found partial products are shown for 27 × 36 and 258 × 17:

South Africa	Belize
27	258
× 36	× 17
810	258
162	1806

Notice in the first example that the first product (27 × 30) was represented with the 0, whereas in the second example, only the number of tens was written (258 × 10 was recorded as 258) but their value was remembered in the alignment of the second partial product.

A woman from the Philippines multiplied 48 × 35 in her head in the following way: 48 × 35 is 50 × 35 minus 2 × 35. Since 50 × 35 is 50 × 30, or 1500, plus 50 × 5, or 250, 50 × 35 equals 1750. This answer must be reduced by the product of 2 × 35, or 70, so the answer to 48 × 35 is 1680.

A woman from Iran and a man from Iraq applied algorithms that are similar to the algorithm generally taught in the United States, with the exception that the numbers to be "carried" are not written in the same location. In the United States, the carried value is written above the column to which the number will subsequently be added, whereas in Iran and Iraq, the numbers to be carried are written off to the side.

Iran		Iraq	
			2
423		755	
× 19		× 5	
3807	2	3775	
423	2		
8037			

People believe that the algorithm they use is easiest

Alternative Algorithms for Division

People all over the world, including many in the United States, use a division algorithm that looks different from the standard algorithm because of the way the numbers are written. In the United States, the typical division algorithm for finding the number of 4s in 260, follows:

$$\begin{array}{r} 65 \\ 4\overline{)260} \\ \underline{-24} \\ 20 \\ \underline{-20} \\ 0 \end{array}$$

A woman from Laos showed how she learned to divide 65 by 2. Notice that the dividend, 65, is placed on the left, and the divisor, 2, is placed on the right, whereas the quotient, 32.5 (written here as 32,5), is placed under the divisor.

$$\begin{array}{r|l} 65 & 2 \\ \underline{3} & \overline{} \\ 05 & 32,5 \\ \underline{-4} & \\ 10 & \\ \underline{-10} & \\ 0 & \end{array}$$

This manner of writing the dividend on the left, the divisor on the right, and the quotient under the divisor was shared by people who learned it in Armenia, Cambodia, Iran, Ireland, Pakistan, Russia, Spain, and Vietnam.

Final Comments

The current mathematics-reform movement in the United States is de-emphasizing the role that procedurally oriented algorithms should play in school. However, in spite of this de-emphasis, students are still taught algorithms by which they are expected to add, subtract, multiply, and divide. This article began with an example of an algorithm invented by a third-grade child who was struggling to make sense of division. Other examples of alternative algorithms have been identified as being used by people from various cultures. Although these people probably did not invent these algorithms, they inherited them as part of their ancestral education. The algorithms we use in school are a matter of convention; they are arbitrary. That is, absolutely nothing is sacred about any of them.

I am neither advocating that teachers teach several different algorithms for a given operation nor suggesting that one algorithm is more "conceptual" than another. I am advocating that teachers allow opportunities for students to present alternative algorithms — whether the students invent them or learn them — and then lead a discussion about the meanings of the operations, with the goal of students' understanding why the algorithm works.

A student in one of my mathematics-methods courses once shared with the class the equal-addition algorithm she had learned for subtraction. After she shared it, many of the students in the class who had been schooled in the traditionally taught algorithm involving "borrowing" expressed their disbelief that their peer would use such a "difficult" algorithm. She replied that her algorithm is not difficult; the one that everyone else is using is difficult. She was quite comfortable with her algorithm and could not figure out how the "borrowing" algorithm worked, which suggests that people have a tendency to believe that the algorithm they use is easiest, regardless of what it is.

It is my hope that teachers not only will become more aware of the diversity of approaches but also might actively seek these approaches among their own students for discussion in their classes. I hope that teachers with students of similar racial and ethnic backgrounds will increase their sensitivity to the many invented algorithms their students might create when learning to operate on numbers and perhaps allow their students a forum for discussing the reasoning that enabled the creation of these different algorithms.

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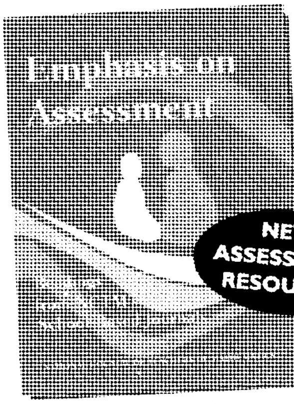

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